Improper Integrals (7.8)

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<u>Review of integrals</u>:

Ex 1)

$$\int_{e^2}^{e^4} \frac{2}{x \cdot \ln x} dx$$ ✽ substitution

$u = \ln(x)$
$dx = xdu$

$$= \int_{\ln(e^2)}^{\ln(e^4)} \frac{2\cancel{x}}{\cancel{x} \cdot u} du$$

$$= \int_2^4 \frac{2}{u} du$$

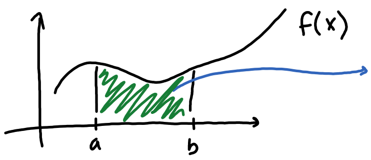
$$= 2 \ln u \Big]_2^4$$

$$= 2 \ln 4 - 2 \ln 2$$
$$= 2 \ln 2$$

recall: ln rules
- $\ln(a^b)$
  $\to b \cdot \ln(a)$
- $\ln(e) = 1$

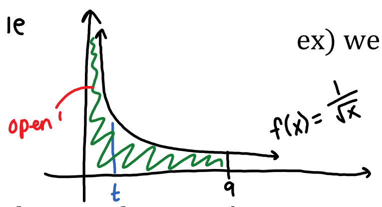
<u>Improper integrals</u>

Until now, $\int_a^b f(x)\,dx$:
- *f(x)* is continuous between *a* and *b*



$\int_a^b f(x)\,dx =$ the area under the curve

NOW: *f* is not necessarily continuous over [a,b] anymore.



ex) we <u>want</u> $\int_0^9 f(x)\,dx$

<u>Work around</u>: we can't put 0 into $f(x)$

<u>Instead</u>: go from t to 9, where $t > 0$.

$\to$ the area from t to 9 is closed and finite.

$$\int_t^9 \frac{1}{\sqrt{x}} dx = \int_t^9 x^{-\frac{1}{2}} dx$$ ✽ power rule

$$= \frac{x^{\frac{1}{2}}}{\frac{1}{2}}\Bigg]_t^9 = 2x^{\frac{1}{2}}\Bigg]_t^9$$

$$= 2(9)^{\frac{1}{2}} - 2t^{\frac{1}{2}}$$
$$= 6 - 2\sqrt{t}$$

<u>recall</u>: power rule

$$\int x^n\,dx$$

$$= \frac{x^{n+1}}{n+1}, \qquad n \neq 1$$

$\to$ this is the area between t and 9 under $\frac{1}{\sqrt{x}}$

**use limit to let t go to 0 and see what happens o the area

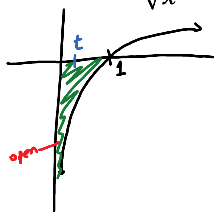
$$\lim_{t \to 0}(6 - 2\sqrt{t})$$
$$= 6$$

Despite the area being apparently open, it is finite!

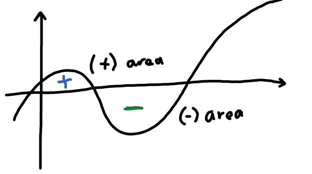
This is an **improper integral** with a discontinuous integrand ($f(x)$ is not defined for all *x* in *[a,b]* or $f(x)$ is not continous on *[a,b]*.

Ex 2)

$$f(x) = \frac{\ln(x)}{\sqrt{x}}, \qquad \int_0^1 f(x)\,dx$$



recall:



Again:

$$\int_t^1 \frac{\ln(x)}{\sqrt{x}}\,dx$$

integration by parts ☆

$f(x) = \ln(x)$	$g'(x) = x^{-\frac{1}{2}}$
$f'(x) = \frac{1}{x}$	$g(x) = 2x^{\frac{1}{2}}$

$$= \ln(x)\cdot 2x^{\frac{1}{2}}\Big]_t^1 - \int_t^1 \frac{2x^{1/2}}{x}\,dx$$

$$= \ln(\mathrm{x})\cdot 2\mathrm{x}^{\frac{1}{2}}\Big]_t^1 - \int_t^1 \frac{2}{\sqrt{x}}\,dx$$

$$= \ln(1)\cdot 2(1)^{\frac{1}{2}} - \ln(\mathrm{t})\cdot 2\mathrm{t}^{\frac{1}{2}} - 2(2(1)^{\frac{1}{2}} - 2\mathrm{t}^{\frac{1}{2}})$$

$$= \underline{-\ln(t)\cdot 2t^{\frac{1}{2}} - 4 + 4\sqrt{t}}$$ ☆ area between t and 1 under $\frac{\ln(x)}{\sqrt{x}}$.

☆ now, take the limit $t \to 0$

$$\lim_{t\to0}(\underbrace{\overset{\to -\infty}{-2\ln(t)}\cdot \overset{\to 0}{t^{\frac{1}{2}}}}_{\text{L hopital}} \overset{\to -4}{-\,4} + \overset{\to 0}{4\sqrt{t}})$$

$$\lim_{t\to0}(-2\ln(t)\cdot t^{\frac{1}{2}})$$ ☆need a fraction

$$\lim_{t\to0} -\frac{2\ln(t)}{\frac{1}{\sqrt{t}}} = \lim_{t\to0}\frac{-\frac{2}{t}}{-\frac{1}{2}t^{-\frac{3}{2}}} = 4\cdot\lim_{t\to0}\frac{t^{\frac{3}{2}}}{t} = 4\cdot\lim_{t\to0}\sqrt{t}\boxed{= 0}$$

Therefore:

$$\lim_{t\to0}(\underset{\to 0}{-2\ln(t)\cdot t^{\frac{1}{2}}} \underset{\to -4}{-\,4} + \underset{\to 0}{4\sqrt{t}}) = -4$$